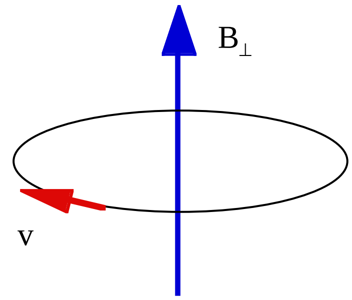
Spin Dynamics

Particle trajectory governed by Lorentz force:



$$\frac{d(\gamma m\vec{v})}{dt} = e\left\{\vec{E} + \vec{v}\times\vec{B}\right\}$$

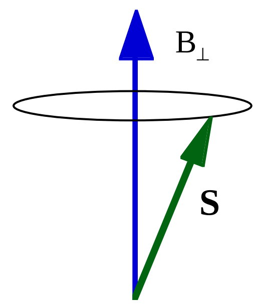
$$\frac{d\vec{v}}{dt} = -\frac{e}{m\gamma}\left\{\vec{B}\times\vec{v}\right\} \qquad \left(for\ \vec{E}=0\right)$$

$$= -\left[\frac{e\vec{B}_\perp}{m\gamma}\right]\times\vec{v}$$

$$= -\vec{\Omega}_{rev}\times\vec{v}$$

(assuming γ changing slowly)

In a frame rotating with the particle velocity vector, the equation for spin precession is:



$$\frac{d\vec{S}}{dt} = -\frac{e}{m\gamma}\left\{G\gamma\vec{B} - (\gamma-1)G\frac{(\vec{v}\cdot\vec{B})\vec{v}}{v^2} + \gamma\left(G - \frac{1}{\gamma^2-1}\right)\frac{\vec{E}\times\vec{v}}{c^2}\right\}\times\vec{S}$$

$$= -\frac{e}{m\gamma}\left\{G\gamma\vec{B}_\perp + G\vec{B}_{\|}\right\}\times\vec{S} \qquad \left(for\ \vec{E}=0\right)$$

$$= -\left[\frac{e\vec{B}_\perp}{m\gamma}G\gamma\right]\times\vec{S} \qquad \left(for\ \vec{E}=0,\quad \vec{B}_{\|}=0\right)$$

$$= -G\gamma\vec{\Omega}_{rev}\times\vec{S}$$

Here, the fields are those in the lab frame, whereas the spin vector is in the particle's rest frame.

For a pure vertical guide field in a circular accelerator, the spin precesses $G\gamma$ times per revolution. Thus, the "spin tune" is $\nu_s = G\gamma$.

Depolarizing Spin Resonances

Field perturbations cause undesired precession, and when resonance conditions hold can generate depolarization...

- Imperfection Resonances
  - arise from sampling of error fields, fields due to closed orbit errors, etc.
  - $G\gamma = integer$

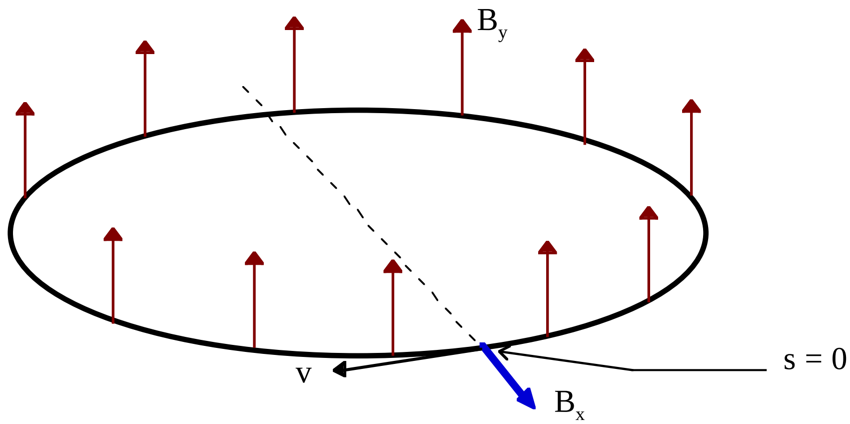
- Intrinsic Resonances
  - arise from sampling of focusing fields due to finite beam emittance
  - $G\gamma \pm \nu_y = integer$;

    ν_y = vertical betatron tune

These resonance conditions can be avoided through the use of "Siberian Snakes."

Crossing Imperfection Resonances...

Consider an accelerator which has perfect vertical guide fields except for a single small horizontal field error at one location (s = 0, say) in the machine. (This is equivalent to a very weak partial Siberian Snake!)



The horizontal field produces a spin rotation about the x-axis of amount $\Delta\phi = 2\pi\varepsilon = G\gamma\, B_x\, L/(B\rho)$, where L is the longitudinal extent of the field error; $(B\rho) = p/e$ is the magnetic rigidity, the momentum per unit charge.

Proceeding once around the accelerator, the spin is governed by

$$\boxed{\begin{aligned}\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_{n+1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \cos 2\pi G\gamma & 0 & \sin 2\pi G\gamma \\ 0 & 1 & 0 \\ -\sin 2\pi G\gamma & 0 & \cos 2\pi G\gamma \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_n \\ &= \begin{pmatrix} \cos 2\pi G\gamma & 0 & \sin 2\pi G\gamma \\ \sin\varepsilon \sin 2\pi G\gamma & \cos\varepsilon & -\sin\varepsilon\cos 2\pi G\gamma \\ -\cos\varepsilon \sin 2\pi G\gamma & \sin\varepsilon & \cos\varepsilon\cos 2\pi G\gamma \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_n\end{aligned}}$$

The stable spin direction at s = 0 is found by finding the total spin rotation angle and spin rotation axis using the resulting matrix above. To do so, we can use the infinitesimal rotation generators

$$\Delta_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Delta_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Delta_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from which the rotation axis of M can be found from

$$n_x \propto trace(\Delta_x M) = trace\begin{pmatrix} 0 & & \\ & -\sin 2\pi\varepsilon & \\ & & -\sin 2\pi\varepsilon \cos 2\pi G\gamma \end{pmatrix} = -\sin 2\pi\varepsilon(1+\cos 2\pi G\gamma)$$

$$n_y \propto trace(\Delta_y M) = trace\begin{pmatrix} -\cos 2\pi\varepsilon \sin 2\pi G\gamma & & \\ & 0 & \\ & & -\sin 2\pi G\gamma \end{pmatrix} = -\sin 2\pi G\gamma(1+\cos 2\pi\varepsilon)$$

$$n_z \propto trace(\Delta_z M) = trace\begin{pmatrix} -\sin 2\pi\varepsilon \sin 2\pi G\gamma & & \\ & 0 & \\ & & 0 \end{pmatrix} = -\sin 2\pi\varepsilon \sin 2\pi G\gamma$$

The proper normalization gives

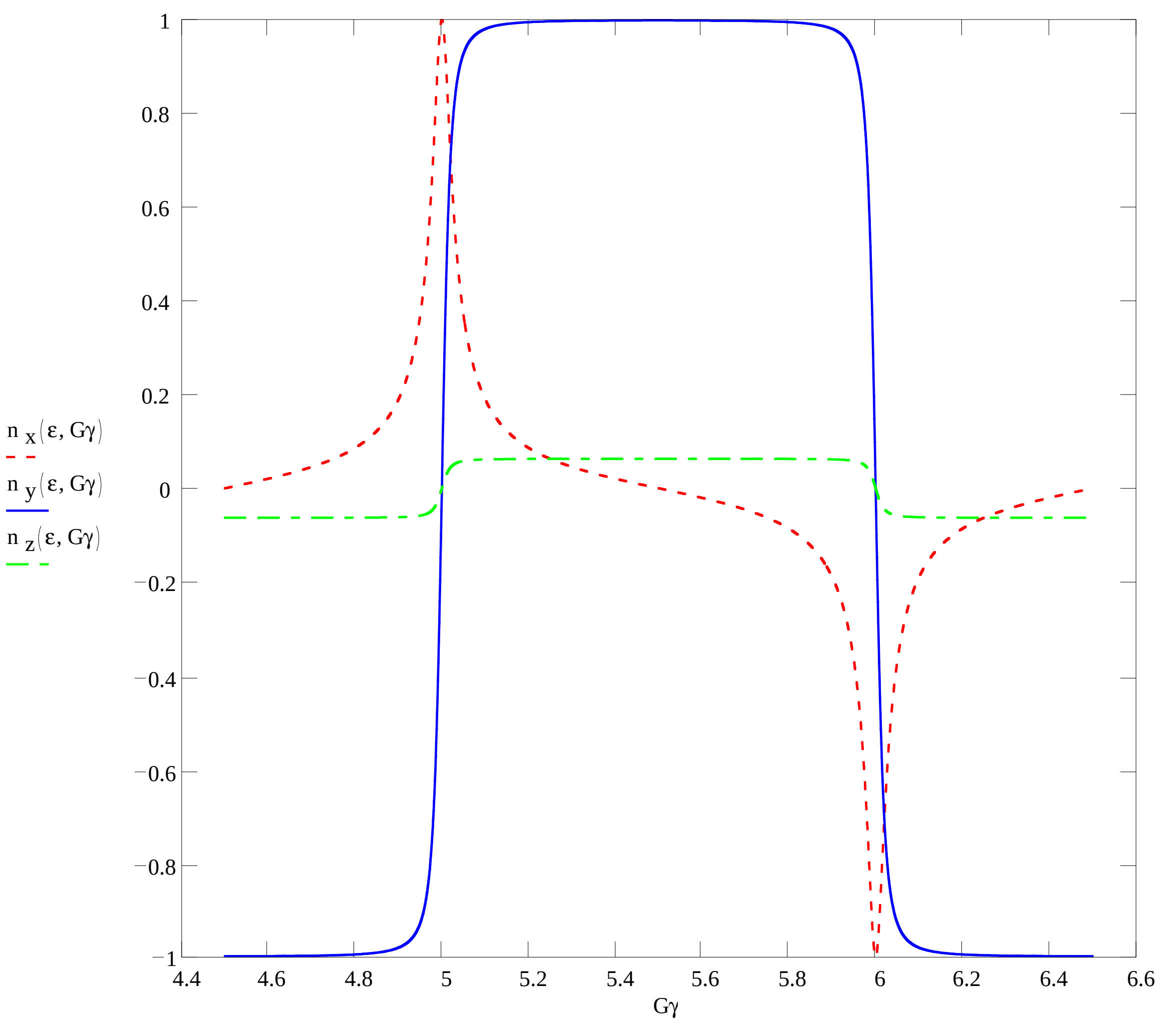
$$\boxed{\begin{aligned} n_x &= -\frac{\sin \pi\varepsilon \cos \pi G\gamma}{\sqrt{\sin^2 \pi\varepsilon + \cos^2 \pi\varepsilon \sin^2 \pi G\gamma}}, \\ n_y &= -\frac{\cos \pi\varepsilon \sin \pi G\gamma}{\sqrt{\sin^2 \pi\varepsilon + \cos^2 \pi\varepsilon \sin^2 \pi G\gamma}}, \\ n_z &= -\frac{\sin \pi\varepsilon \sin \pi G\gamma}{\sqrt{\sin^2 \pi\varepsilon + \cos^2 \pi\varepsilon \sin^2 \pi G\gamma}}. \end{aligned}}$$

We see immediately that if $\varepsilon = 0$, then $\boldsymbol{n} = (0, \pm1, 0)$

and if $G\gamma =$ *integer*, then $\boldsymbol{n} = (\pm1, 0, 0)$.

From the above, a small horizontal perturbing field changes the stable spin direction as a function of energy. At the field error, the stable spin direction is vertical (or nearly so) far from resonance. As resonance ($G\gamma$ = *integer*) is approached, the stable spin direction moves toward the horizontal plane. Exactly on resonance, the stable spin direction is in the horizontal plane and particle spin precesses about a horizontal axis as intuitively expected. Below is a plot of the stable spin direction components as a function of $G\gamma$ for ε= 0.02...

$\varepsilon := 0.02$ $\qquad \Delta\phi := 2\cdot\pi\cdot\varepsilon$ $\qquad \Delta\phi = 7.2\cdot\mathrm{deg}$



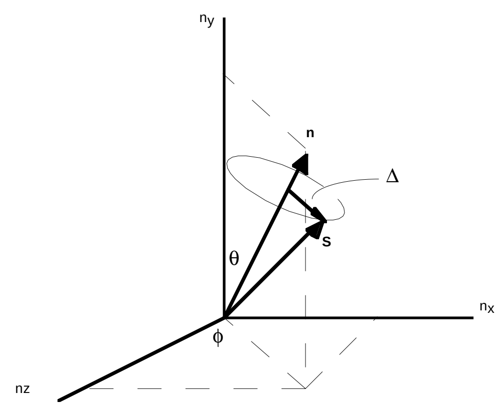
Adiabatic Spin Flip

For a process to be considered adiabatic, the slowly varying parameter of the system, λ, must only change by a small amount during the characteristic time scale, T_0, of the periodic motion. That is,

$$\Delta\lambda = \frac{d\lambda}{dt} T_0 << \lambda$$

For our case, we consider that the direction of the stable spin direction is changing as we cross $G\gamma = k =$ *integer*:

- Consider θ (= $\pi/2$ at $G\gamma = k$) as the slowly varying parameter
- Define $\alpha = d(G\gamma)/d\theta_{orbit} = \Delta(G\gamma)$ per turn$/2\pi$ = "acceleration rate"
- $n_y = \cos\theta$, as shown previously, varies with $G\gamma$
- $T_0 = T_s = 1/(f_{rev}\,\nu_s) = 1/(f_{rev}\,\varepsilon)$ near $G\gamma = k$



Then,

$$\Delta\theta = \frac{d\theta}{dt} T_S = \frac{d\theta}{dn_y}\frac{dn_y}{dG\gamma}\frac{dG\gamma}{d\theta_{orbit}}\frac{d\theta_{orbit}}{dt}\frac{1}{f_{rev}\varepsilon} << \frac{\pi}{2}$$

which reduces to...

$$\Delta\theta = -\frac{2\pi\alpha}{\varepsilon}\frac{\sqrt{\sin^2\pi\varepsilon + \cos^2\pi\varepsilon\sin^2\pi G\gamma}}{\sin\pi\varepsilon}\frac{dn_y}{dG\gamma}$$

$$= \frac{2\pi\alpha}{\varepsilon}\frac{1}{\sin\pi\varepsilon}\frac{\pi\cos\pi G\gamma\cos\pi\varepsilon\sin^2\pi\varepsilon}{\left(\sqrt{\sin^2\pi\varepsilon + \cos^2\pi\varepsilon\sin^2\pi G\gamma}\right)^2} << \frac{\pi}{2}$$

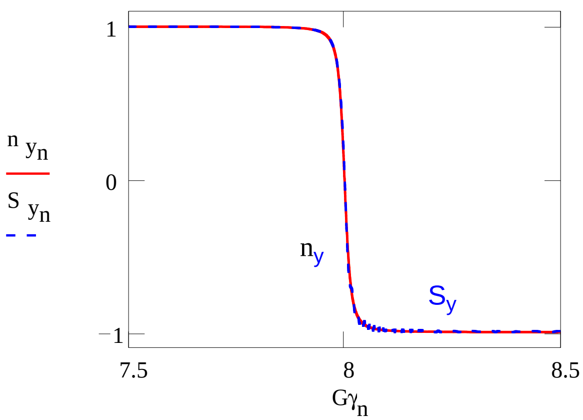
As Gγ $\longrightarrow$ integer and since ε is small, the above reduces to a condition for **adiabaticity**...

$$\frac{4\alpha}{\varepsilon^2} << 1$$

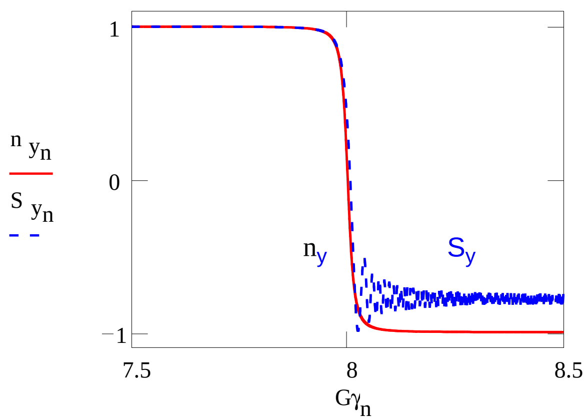
So, as long as the change in Gγ per turn is much less than the square of the resonance strength, then the motion will be adiabatic and the spin precessions will closely follow the changing direction of the stable spin vector.

Some examples of crossing a resonance of strength ε = 0.015 (ε^2 = 2.25x10^{-4}) at various speeds...

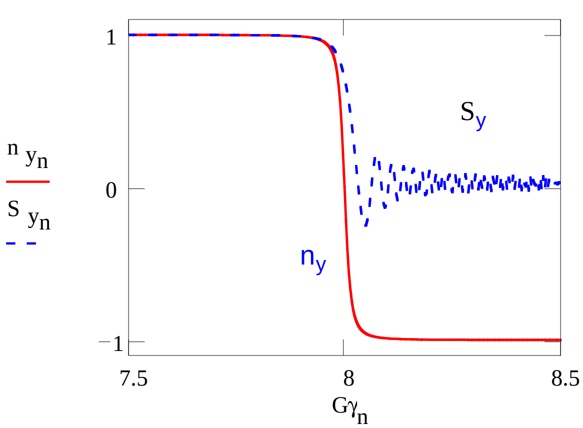
α = 0.53x10^{-4}



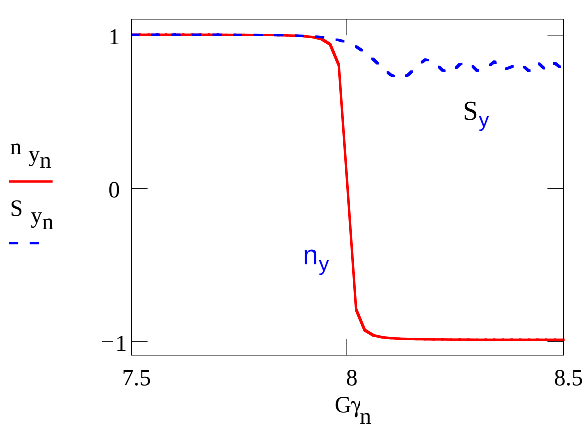
α = 1.6x10^{-4}



α = 5.3x10^{-4}



α = 16x10^{-4}



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Remarks:

- Passing through the resonance, the stable spin direction changes sign -- “spin flip.”
- So long as the resonance is passed adiabatically, the particles will follow the stable spin direction and polarization of the beam will be preserved.
- If the resonance is passed very quickly, then the stable spin direction can change sign quickly enough that the spin simply begins precessing about the new direction; again polarization would be preserved.
- In intermediate cases, where either the crossing is neither quick nor adiabatic, the spin precession of the particles will not follow the change of the stable spin axis, and, because there is an inherent spread in precession frequency of the particles, beam depolarization will result. The resulting depolarization can be estimated using the Froissart-Stora formula.

Froissart-Stora Formula

Let α characterize the rate at which the resonance is crossed:

$$\boxed{\alpha \equiv \frac{d(G\gamma)}{d\theta} = \frac{G}{2\pi f_{rev}} \frac{d\gamma}{dt}}$$

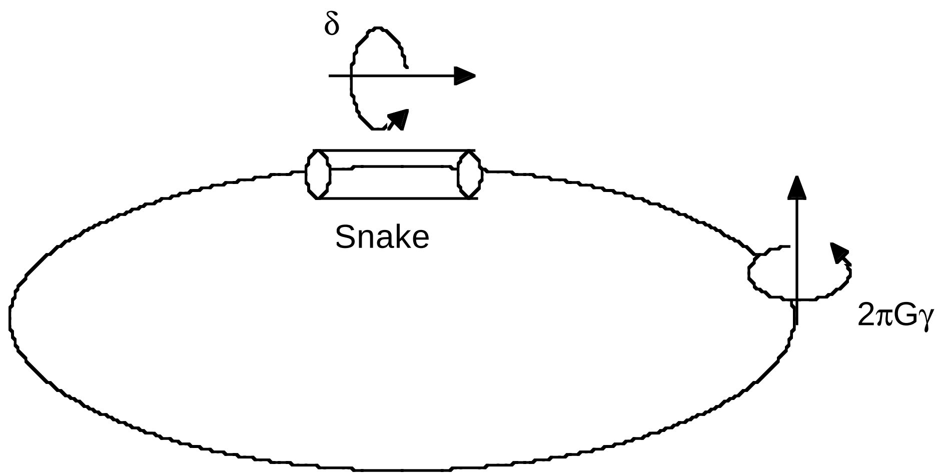
Consider a particle whose initial spin component along the stable spin direction is P_i . After crossing the resonance, its spin component P_f along the resulting stable spin direction is given by the Froissart-Stora formula:

$$\boxed{\frac{P_f}{P_i} = 2e^{-\pi\frac{|\varepsilon|^2}{2\alpha}} - 1}$$

Comment on intrinsic resonances and spin flippers:

The above discussion describes the case of a spin imperfection resonance. Intrinsic resonances, or artificially induced resonances from "spin flippers," are essentially the same, except that the horizontal field error is not constant. After a transformation into a reference frame that rotates at the frequency of the horizontal field, these resonances can be understood in the same way as imperfection resonances.

Siberian Snakes and Partial Siberian Snakes...



Rotation of the spin vector upon passing once around the accelerator is governed by

$$\boxed{\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 2\pi G\gamma & 0 & \sin 2\pi G\gamma \\ 0 & 1 & 0 \\ -\sin 2\pi G\gamma & 0 & \cos 2\pi G\gamma \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_n = M \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}_n}$$

The above rotation transformation can be characterized by a rotation through an angle about a particular axis. The rotation angle will be 2π times the “spin tune”and is derived from the trace of M.

And so,

$$\boxed{\begin{aligned} M &= \begin{pmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 2\pi G\gamma & 0 & \sin 2\pi G\gamma \\ 0 & 1 & 0 \\ -\sin 2\pi G\gamma & 0 & \cos 2\pi G\gamma \end{pmatrix} \\ &= \begin{pmatrix} \cos\delta\cos 2\pi G\gamma & \sin\delta & \cos\delta\sin 2\pi G\gamma \\ \sin\delta\cos 2\pi G\gamma & \cos\delta & -\sin\delta\sin 2\pi G\gamma \\ -\sin 2\pi G\gamma & 0 & \cos 2\pi G\gamma \end{pmatrix} \end{aligned}}$$

From which

$$\mathrm{Tr}M = 1 + 2\cos 2\pi\nu_s \qquad \text{(for any rotation matrix)}$$

$$= \cos\delta\cos 2\pi G\gamma + \cos\delta + \cos 2\pi G\gamma$$

$$= (1+\cos\delta)(1+\cos 2\pi G\gamma) - 1$$

or,

$$2(1 + \cos 2\pi\nu_s) = (1+\cos\delta)(1+\cos 2\pi G\gamma)$$

and thus,

$$\cos\pi\nu_s = \cos(\delta/2)\,\cos\pi G\gamma$$

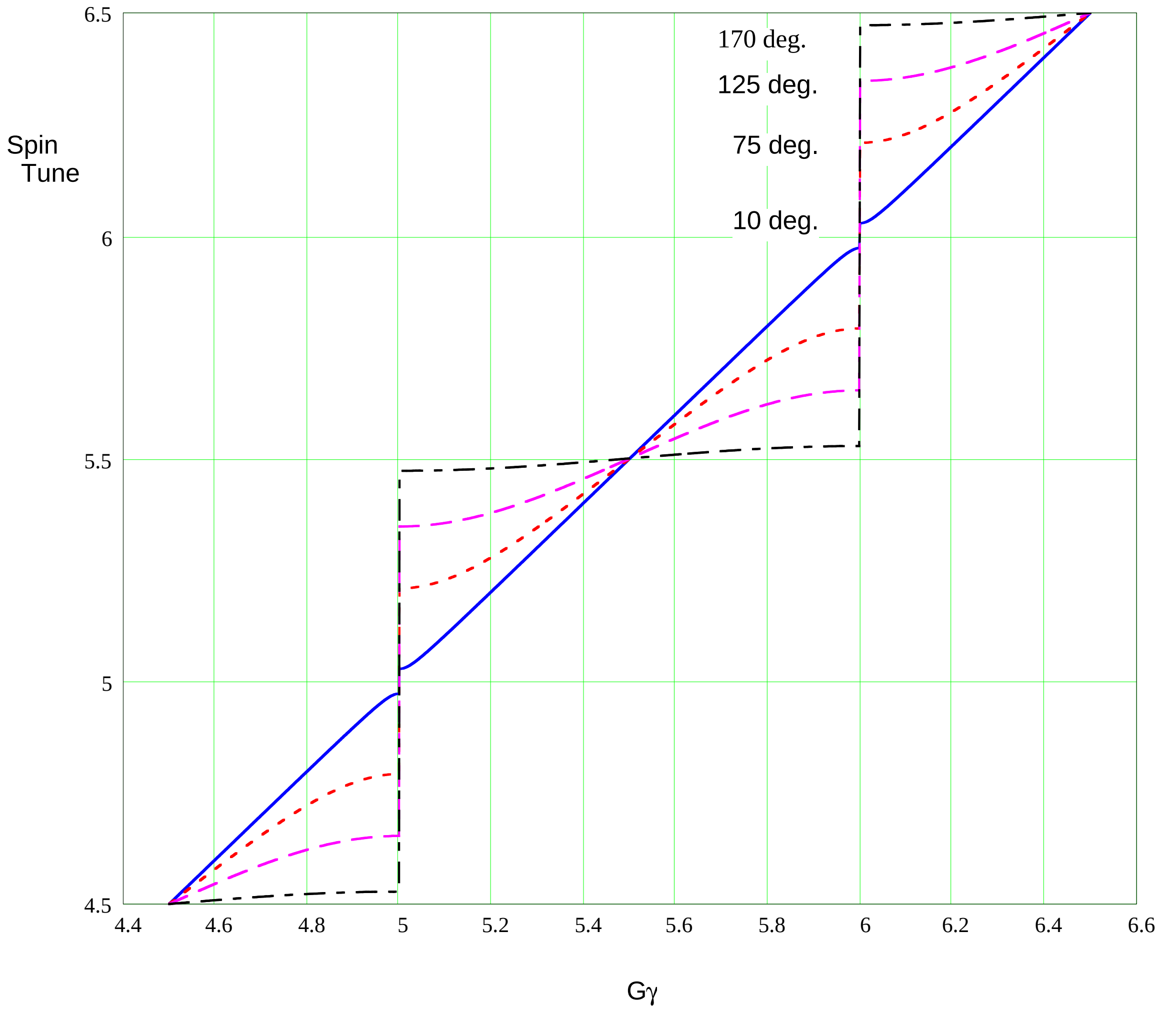
So, if $\delta = \pi$ (Full Siberian Snake), then $\nu_s = 1/2$,

if $\delta = 0$ (No Snake!), then $\nu_s = G\gamma$,

and otherwise (Partial Snake) , then

ν_s cannot be an integer!

Plot Spin Tune versus Gγ for various values of δ...



Spin tune versus Gγ for values of Snake rotation angles 10, 75, 125, and 170 degrees.

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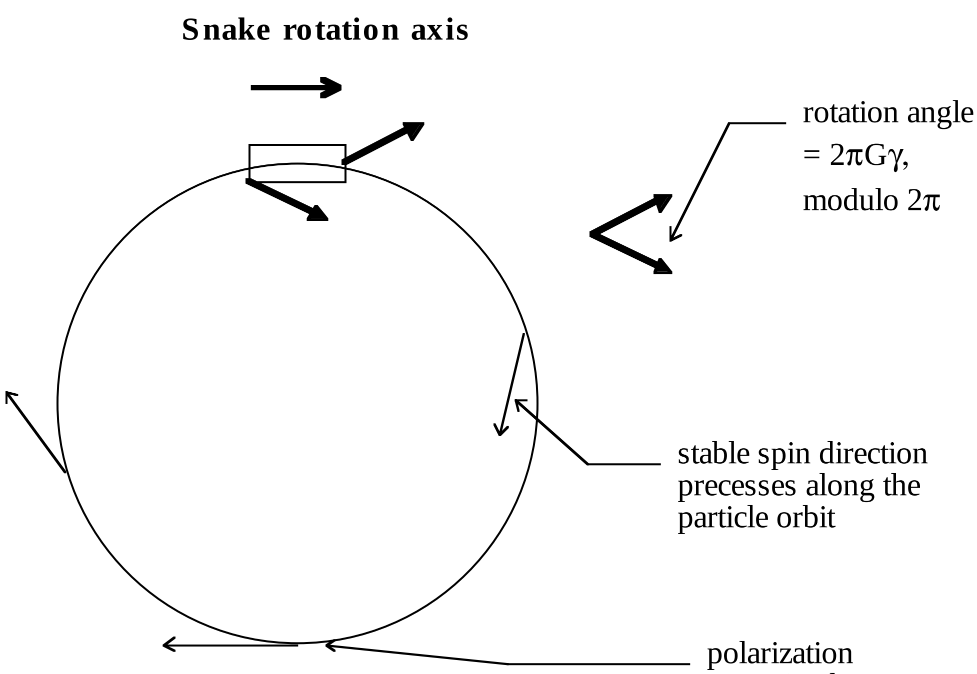
Stable Spin Direction

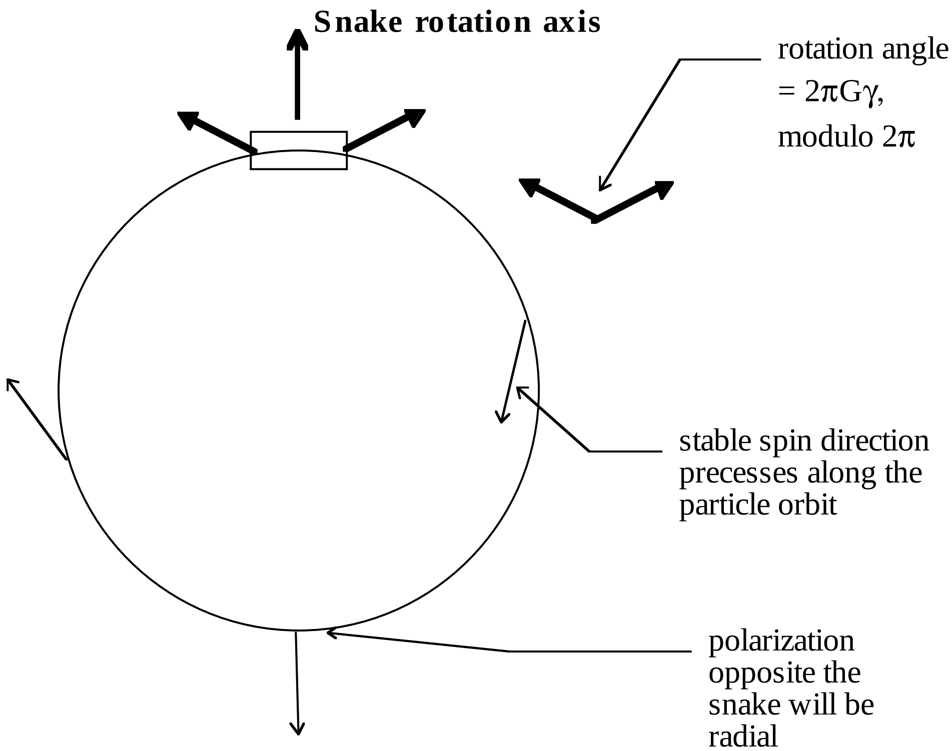
Just as there is a closed orbit in an accelerator, about which all particles undergo betatron oscillations, there is a closed spin orientation -- the stable spin direction -- which is a function of the accelerator lattice, and about which all particles undergo spin precession.

For a uniform vertical magnetic field, the stable spin direction is simply vertical -- particles will precess about the vertical direction with frequency given by the spin tune, $G\gamma$ for this case.

For the case of a single Siberian Snake as described above, the stable spin orientation about the circumference of the accelerator is illustrated here:

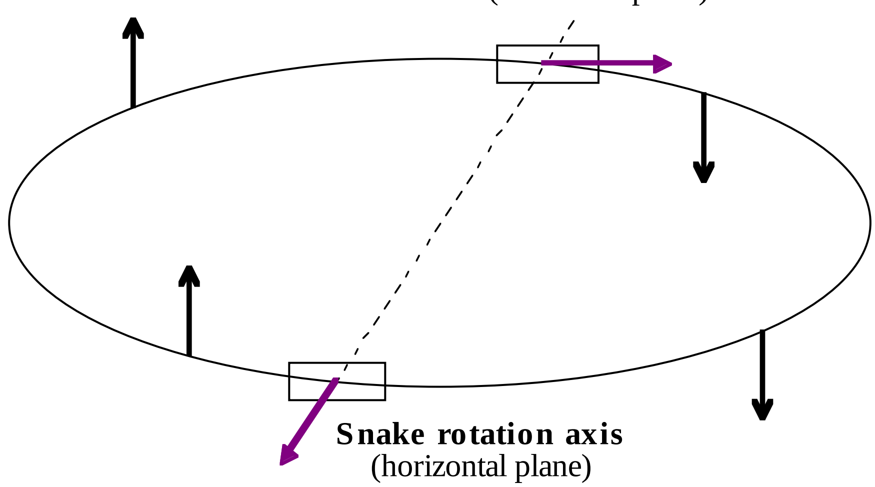


To have transverse polarization opposite the Snake, say, one could envision a Snake which rotates the spin by 180 degrees about the radial transverse axis. The spin tune would still be 1/2, but the stable spin direction opposite the Snake would be radial:



In the above two cases, the stable spin direction at any one location in the accelerator (other than opposite the Snake) depends upon the particle energy. Introducing two Snakes into the accelerator, with their rotation axes in the horizontal plane and 90 degrees apart, the Snake condition can be met and at the same time the stable spin direction will be vertical (up or down) independent of energy...

Two Snakes...



For this case, the spin matrix for the ring will be:

$$
\begin{aligned}
M &= M_2 M_1 \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{pmatrix}\right]\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{pmatrix}\right] \\
&= \begin{pmatrix} C & 0 & S \\ 0 & -1 & 0 \\ S & 0 & -C \end{pmatrix}\begin{pmatrix} -C & 0 & -S \\ 0 & -1 & 0 \\ -S & 0 & C \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

with trace = -1, yielding ν_s = 1/2.

In RHIC, the axes of the two Snakes are 90 degrees apart, but each ±45 degrees from the longitudinal axis. Thus, the two magnetic devices can be mechanically the same, operating at the same strengths, but of opposite electrical polarities.